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# Higher order perturbation expansion of waves in water of variable depth

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ABSTRACT

In this work, we extended the application of "the modified reductive perturbation method" to long waves in water of variable depth and obtained a set of KdV equations as the governing equations. Seeking a localized travelling wave solution to these evolution equations we determine the scale function $c_1(\tau)$ so as to remove the possible secularities that might occur. We showed that for waves in water of variable depth, the phase function is not linear anymore in the variables $x$ and $t$. It is further shown that, due to the variable depth of the water, the speed of the propagation is also variable in the $x$ coordinate.



## 1. Introduction

In collisionless cold plasma, in fluid-filled elastic tubes and in shallow-water waves, due to nonlinearity of the governing equations, for weakly dispersive case one obtains the Korteweg–de Vries (KdV) equation for the lowest order term in the perturbation expansion, the solution of which may be described by solitons [1]. To study the higher order terms in the perturbation expansion, the reductive perturbation method has been introduced by use of the stretched time and space variables [2]. However, in such an approach some secular terms appear which can be eliminated by introducing some slow scale variables [3] or by a renormalization procedure of the velocity of the KdV soliton [4]. Nevertheless, this approach remains somewhat artificial, since there is no reasonable connection between the smallness parameters of the stretched variables and the one used in the perturbation expansion of the field variables. The choice of the former parameter is based on the linear wave analysis of the concerned problem and the wave number or the frequency is taken as the perturbation parameter [5]. On the other hand, at the lowest order, the amplitude and the width of the wave are expressed in terms of the unknown perturbed velocity, which is also used as the smallness parameter. This causes some ambiguity over the correction terms. Another attempt to remove such secularities was made by Kraenkel et al. [6] for long water waves by use of the multiple timescale expansion but could not obtain explicitly the correction terms to the wave speed.

In order to remove these uncertainties, Malfliet and Wieers [7] presented a dressed solitary wave approach, which is based on the assumption that the field variables admit localized travelling wave solution. Then, for the long wave limit, they expanded the field variables and the wave speed into a power series of the wave number, which is assumed to be the only smallness parameter, and obtained the explicit solution for various order terms in the expansion. However, this approach can only be used when one studies the progressive wave solution to the original nonlinear equations and it does not give any idea about the form of evolution equations governing the various order terms in the perturbation expansion. In our previous paper [8], we have presented a method called "the modified reductive perturbation method" to examine the contributions of higher order terms in the perturbation expansion and applied it to weakly dispersive ion-acoustic plasma waves, solitary waves in a fluid-filled elastic tube [9] and long water waves of constant depth [10]. In these works, we have shown that the lowest order term in the perturbation expansion is governed by the nonlinear Korteweg–de Vries equation, whereas the higher order terms in the expansion are governed by the degenerate Korteweg–de Vries equation with the non-homogeneous term. By employing the hyperbolic tangent method a progressive wave type of solution was sought and the

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possible secularities were removed by selecting the scaling parameter in a special way. The basic idea in this method was the inclusion of higher order dispersive effects through the introduction of the scaling parameter $c$, to balance the higher order nonlinearities with dispersion. The negligence of higher order dispersive effects in the classical reductive perturbation method leads to the imbalance between the nonlinearity and the dispersion, which resulted in some secular terms in the solution of evolution equations. As a matter of fact, the renormalization method presented by Kodama and Taniuti [4] is different but rather involved formulation of the same idea.

In this work, we extended the application of "the modified reductive perturbation method" to long waves in water of variable depth and obtained a set of KdV equations as the governing equations. Seeking a localized travelling wave solution to these evolution equations we determined the scale function $c_1(\tau)$ so as to remove the possible secularities that might occur. We showed that for waves in water of variable depth, the phase function is not linear anymore in the variables $x$ and $t$. It is further shown that, due to the variable depth of the water, the speed of the propagation is also variable in the $x$ coordinate.

## 2. Modified reductive perturbation formalism for water waves

We consider a two-dimensional incompressible inviscid fluid in a constant gravitational field $g$. The space coordinates are denoted by $(x, z)$ and the corresponding velocity components by $(u, w)$. The gravitational acceleration is in negative $z$ direction. The equations describing such a fluid are:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad \text{(incompressibility)} \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0, \tag{3}$$

where $\rho$ is the mass density and $p$ is the fluid pressure function. Assuming that the flow is irrotational, the velocity vector can be derived from a scalar potential $\hat{\phi}(x, z, t)$ as

$$u = \frac{\partial \hat{\phi}}{\partial x}, \quad w = \frac{\partial \hat{\phi}}{\partial z}. \tag{4}$$

Then, the incompressibility condition reduces to

$$\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial z^2} = 0, \tag{5}$$

and the Euler equations become

$$\frac{p - p_0}{\rho} = -\frac{\partial \hat{\phi}}{\partial t} - \frac{1}{2}\left[\left(\frac{\partial \hat{\phi}}{\partial x}\right)^2 + \left(\frac{\partial \hat{\phi}}{\partial z}\right)^2\right] - gz, \tag{6}$$

where $p_0$ is an integration constant.

We consider the case of a fluid of height $h(x)$, Fig. 1, bounded above by a steady atmosphere with pressure $p_0$. Let the upper surface be described by $z = \hat{\psi}(x, t)$. The kinematic boundary condition on this surface can be expressed as:

$$\frac{\partial \hat{\phi}}{\partial z} = \frac{\partial \hat{\psi}}{\partial t} + \frac{\partial \hat{\phi}}{\partial x}\frac{\partial \hat{\psi}}{\partial x}, \quad \text{on } z = \hat{\psi}(x, t). \tag{7}$$

From Eq. (6), the dynamical boundary condition on this surface reads

$$\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \hat{\phi}}{\partial x}\right)^2 + \left(\frac{\partial \hat{\phi}}{\partial z}\right)^2\right] + g\hat{\psi} = 0, \quad \text{on } z = \hat{\psi}(x, t). \tag{8}$$

Finally, the lower boundary is supposed to be rigid. Therefore, at $z = -h_0 + f(x)$, the normal component of the velocity must vanish, *i.e.*,

$$\frac{\partial \hat{\phi}}{\partial z} - \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\partial \hat{\phi}}{\partial x} = 0 \quad \text{at } z = -h_0 + f(x) \tag{9}$$

where $f(x)$ is the profile function of the bottom of the channel.

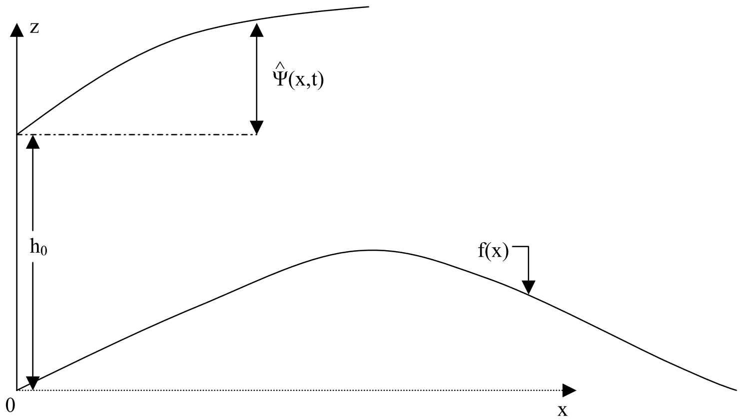


**Fig. 1.** Geometry of the general wave propagation problem.

Now, we shall consider the long wave in shallow-water approximation to the above equations by extending the modified reductive perturbation method developed by us [8] so as to account the variable depth. We assume that the water is coming from a large reservoir or a source so that the problem is of the boundary value type. Extending the proposed method in [8], we introduce the following coordinate stretching

$$\epsilon^{1/2}(x - c_0 t) = \xi - \int_0^\tau c(s)\mathrm{d}s, \quad \tau = \epsilon^{3/2}x, \tag{10}$$

where $\epsilon$ is a small parameter characterizing the smallness of certain physical entities, $c_0$ is a constant and $c(\tau)$ is a scale function to be determined from the solution. Then, the following differential relations hold true

$$\frac{\partial}{\partial x} = \epsilon^{1/2}[1 + \epsilon\, c(\tau)]\frac{\partial}{\partial \xi} + \epsilon^{3/2}\frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial t} = -\epsilon^{1/2}c_0\frac{\partial}{\partial \xi}. \tag{11}$$

For our future purposes we introduce the following new dependent variables

$$\hat{\phi} = \epsilon^{1/2}\phi, \qquad \hat{\psi} = \epsilon\psi. \tag{12}$$

Introducing (11) and (12) into Eqs. (5)–(9) we obtain

$$\frac{\partial^2\phi}{\partial z^2} + \epsilon\frac{\partial^2\phi}{\partial \xi^2} + 2\epsilon^2\left(c\frac{\partial^2\phi}{\partial \xi^2} + \frac{\partial^2\phi}{\partial\xi\partial\tau}\right) + \epsilon^3\left(c^2\frac{\partial^2\phi}{\partial \xi^2} + 2c\frac{\partial^2\phi}{\partial\xi\partial\tau} + \frac{\mathrm{d}c}{\mathrm{d}\tau}\frac{\partial\phi}{\partial\xi} + \frac{\partial^2\phi}{\partial\tau^2}\right) = 0, \tag{13}$$

$$\frac{\partial\phi}{\partial z} = -\epsilon c_0\frac{\partial\psi}{\partial\xi} + \epsilon^2\left[(1+\epsilon c)\frac{\partial\phi}{\partial\xi} + \epsilon\frac{\partial\phi}{\partial\tau}\right]\left[(1+\epsilon c)\frac{\partial\psi}{\partial\xi} + \epsilon\frac{\partial\psi}{\partial\tau}\right], \quad \text{on } z = \epsilon\psi, \tag{14}$$

$$-c_0\frac{\partial\phi}{\partial\xi} + \frac{1}{2}\epsilon\left[(1+\epsilon c)\frac{\partial\phi}{\partial\xi} + \epsilon\frac{\partial\phi}{\partial\tau}\right]^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 + g\psi = 0, \quad \text{on } z = \epsilon\psi, \tag{15}$$

$$\frac{\partial\phi}{\partial z} - \epsilon^2\frac{\mathrm{d}f}{\mathrm{d}\tau}\left[(1+\epsilon c)\frac{\partial\phi}{\partial\xi} + \epsilon\frac{\partial\phi}{\partial\tau}\right] = 0, \quad \text{at } z = -h_0 + f(x). \tag{16}$$

Solving $x$ in terms of $\tau$, $x = \epsilon^{-3/2}\tau$, and introducing it into the expression of $f(x)$ we get $f(x) = f(\epsilon^{-3/2}\tau)$. In order to take the effect of the bump into account we must have $f = O(\epsilon^{5/2})$.

Now, we expand the field quantities into a suitable power series of $\epsilon$ as:

$$\begin{aligned}
&\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \cdots.,\\
&\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \cdots.,\\
&c(\tau) = \epsilon c_1(\tau) + \epsilon^2 c_2(\tau) + \epsilon^3 c_3(\tau) + \cdots.,\\
&f(\tau) = \epsilon^2 h(\tau).
\end{aligned} \tag{17}$$

Introducing (17) into the Eqs. (13)–(16) and setting the coefficients of like powers of $\epsilon$ equal to zero, we obtain the following sets of differential equations and the associated boundary conditions:

$O(1)$ equations

$$\frac{\partial^2\phi_0}{\partial z^2} = 0, \tag{18}$$

and the boundary conditions

$$\frac{\partial \phi_0}{\partial z}\bigg|_{z=0} = 0, \qquad \frac{\partial \phi_0}{\partial z} = 0, \quad \text{at } z = -h_0,$$
$$\left[ -c_0 \frac{\partial \phi_0}{\partial \xi} + \frac{1}{2}\left(\frac{\partial \phi_0}{\partial z}\right)^2 \right]\bigg|_{z=0} + g\psi_0 = 0. \tag{19}$$

$O(\epsilon)$ equations

$$\frac{\partial^2 \phi_1}{\partial z^2} + \frac{\partial^2 \phi_0}{\partial \xi^2} = 0, \tag{20}$$

and the boundary conditions

$$\frac{\partial \phi_1}{\partial z} = 0 \quad \text{at } z = -h_0,$$
$$\left[ \frac{\partial \phi_1}{\partial z} + \psi_0 \frac{\partial^2 \phi_0}{\partial z^2} \right]\bigg|_{z=0} + c_0 \frac{\partial \psi_0}{\partial \xi} = 0, \qquad \left[ -c_0 \frac{\partial \phi_1}{\partial \xi} + \frac{1}{2}\left(\frac{\partial \phi_0}{\partial \xi}\right)^2 \right]\bigg|_{z=0} + g\psi_1 = 0. \tag{21}$$

$O(\epsilon^2)$ equations

$$\frac{\partial^2 \phi_2}{\partial z^2} + \frac{\partial^2 \phi_1}{\partial \xi^2} + 2\frac{\partial^2 \phi_0}{\partial \xi \partial \tau} = 0, \tag{22}$$

and the boundary conditions

$$\frac{\partial \phi_2}{\partial z} = 0 \quad \text{at } z = -h_0,$$
$$\left[ \frac{\partial \phi_2}{\partial z} + \psi_0 \frac{\partial^2 \phi_1}{\partial z^2} - \frac{\partial \phi_0}{\partial \xi}\frac{\partial \psi_0}{\partial \xi} \right]\bigg|_{z=0} + c_0 \frac{\partial \psi_1}{\partial \xi} = 0,$$
$$\left[ -c_0 \left( \frac{\partial \phi_2}{\partial \xi} + \psi_0 \frac{\partial^2 \phi_1}{\partial z \partial \xi} \right) + \frac{\partial \phi_0}{\partial \xi}\left( \frac{\partial \phi_1}{\partial \xi} + \frac{\partial \phi_0}{\partial \tau} \right) + \frac{1}{2}\left(\frac{\partial \phi_1}{\partial z}\right)^2 \right]\bigg|_{z=0} + g\psi_2 = 0. \tag{23}$$

$O(\epsilon^3)$ equations

$$\frac{\partial^2 \phi_3}{\partial z^2} + \frac{\partial^2 \phi_2}{\partial \xi^2} + 2c_1(\tau)\frac{\partial^2 \phi_0}{\partial \xi^2} + 2\frac{\partial^2 \phi_1}{\partial \xi \partial \tau} + \frac{\partial^2 \phi_0}{\partial \tau^2} = 0, \tag{24}$$

and the boundary conditions

$$\left[ \frac{\partial \phi_3}{\partial z} + h(\tau)\frac{\partial^2 \phi_1}{\partial z^2} \right] = 0, \quad \text{at } z = -h_0,$$
$$\left[ \frac{\partial \phi_3}{\partial z} + \psi_0 \frac{\partial^2 \phi_2}{\partial z^2} + \psi_1 \frac{\partial^2 \phi_1}{\partial z^2} - \frac{\partial \psi_0}{\partial \xi}\left( \frac{\partial \phi_0}{\partial \tau} + \frac{\partial \phi_1}{\partial \xi} \right) - \frac{\partial \phi_0}{\partial \xi}\left( \frac{\partial \psi_1}{\partial \xi} + \frac{\partial \psi_0}{\partial \tau} \right) \right]\bigg|_{z=0} + c_0 \frac{\partial \psi_2}{\partial \xi} = 0,$$
$$\begin{aligned}&\Bigg[ -c_0 \left( \frac{\partial \phi_3}{\partial \xi} + \psi_0 \frac{\partial^2 \phi_2}{\partial z \partial \xi} + \frac{1}{2}\psi_0^2 \frac{\partial^3 \phi_1}{\partial z^2 \partial \xi} + \psi_1 \frac{\partial^2 \phi_1}{\partial z \partial \xi} \right) + \frac{1}{2}\left( \frac{\partial \phi_1}{\partial \xi} + \frac{\partial \phi_0}{\partial \tau} \right)^2 \\ &+ \frac{\partial \phi_0}{\partial \xi}\left( \frac{\partial \phi_2}{\partial \xi} + \psi_0 \frac{\partial^2 \phi_1}{\partial z \partial \xi} + c_1(\tau)\frac{\partial \phi_0}{\partial \xi} + \frac{\partial \phi_1}{\partial \tau} \right) + \frac{\partial \phi_1}{\partial z}\left( \frac{\partial \phi_2}{\partial z} + \psi_0 \frac{\partial^2 \phi_1}{\partial z^2} \right) \Bigg]\Bigg|_{z=0} + g\psi_3 = 0.\end{aligned} \tag{25}$$

### *2.1. Solution of the field equations*

From the solution of the set (18) and the associated boundary conditions (19) we have

$$\phi_0 = \varphi(\xi, \tau), \qquad \psi_0 = \frac{c_0}{g}\frac{\partial \varphi}{\partial \xi}, \tag{26}$$

where $\varphi(\xi, \tau)$ is an unknown function of its argument whose governing equation will be obtained later.

To obtain the solution to $O(\epsilon)$ equations given in (20), we introduce the solution (26) into the differential equation (20) and the associated boundary conditions (21), which yields

$$\frac{\partial^2\phi_1}{\partial z^2}+\frac{\partial^2\varphi}{\partial\xi^2}=0,\qquad \left.\frac{\partial\phi_1}{\partial z}\right|_{z=0}+\frac{c_0^2}{g}\frac{\partial^2\varphi}{\partial\xi^2}=0,$$
$$-c_0\left.\frac{\partial\phi_1}{\partial\xi}\right|_{z=0}+\frac{1}{2}\left(\frac{\partial\varphi}{\partial\xi}\right)^2+g\psi_1=0,\qquad \left.\frac{\partial\phi_1}{\partial z}\right|_{z=-h_0}=0. \tag{27}$$

The solution of Eq. (27) along with the use of the boundary conditions yields

$$\phi_1=-\frac{1}{2}\frac{\partial^2\varphi}{\partial\xi^2}(z^2+2h_0z)+\varphi_1(\xi,\tau),$$
$$\psi_1=\frac{c_0}{g}\frac{\partial\varphi_1}{\partial\xi}-\frac{1}{2g}\left(\frac{\partial\varphi}{\partial\xi}\right)^2,\quad c_0=(gh_0)^{1/2}, \tag{28}$$

where $\varphi_1(\xi,\tau)$ is another unknown function whose governing equation will be obtained from the higher order expansions.

To obtain the solution for $O(\epsilon^2)$ equations given in (22) and (23), we introduce the solutions (26) and (28) into Eqs. (22) and (23), which results in

$$\frac{\partial^2\phi_2}{\partial z^2}-\frac{1}{2}\frac{\partial^4\varphi}{\partial\xi^4}(z^2+2h_0z)+2\frac{\partial^2\varphi}{\partial\xi\partial\tau}+\frac{\partial^2\varphi_1}{\partial\xi^2}=0,$$
$$\left.\frac{\partial\phi_2}{\partial z}\right|_{z=0}-3\frac{c_0}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi^2}+h_0\frac{\partial^2\varphi_1}{\partial\xi^2}=0,$$
$$-c_0\left.\frac{\partial\phi_2}{\partial\xi}\right|_{z=0}+h_0^2\frac{\partial\varphi}{\partial\xi}\frac{\partial^3\varphi}{\partial\xi^3}+\frac{\partial\varphi}{\partial\xi}\frac{\partial\varphi_1}{\partial\xi}+\frac{\partial\varphi}{\partial\xi}\frac{\partial\varphi}{\partial\tau}+\frac{h_0^2}{2}\left(\frac{\partial^2\varphi}{\partial\xi^2}\right)^2+g\psi_2=0,$$
$$\left.\frac{\partial\phi_2}{\partial z}\right|_{z=-h_0}=0. \tag{29}$$

The solution of Eq. (29) along with the use of the boundary conditions yields

$$\phi_2=\frac{1}{24}\frac{\partial^4\varphi}{\partial\xi^4}(z^4+4h_0z^3)-\frac{1}{2}\frac{\partial^2\varphi_1}{\partial\xi^2}(z^2+2h_0z)-\frac{\partial^2\varphi}{\partial\xi\partial\tau}z^2+3\frac{c_0}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi^2}z+\varphi_2(\xi,\tau),$$
$$\psi_2=\frac{c_0}{g}\frac{\partial\varphi_2}{\partial\xi}-\frac{h_0^2}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^3\varphi}{\partial\xi^3}-\frac{1}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial\varphi}{\partial\tau}-\frac{1}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial\varphi_1}{\partial\xi}-\frac{h_0^2}{2g}\left(\frac{\partial^2\varphi}{\partial\xi^2}\right)^2,$$
$$\frac{h_0^3}{3}\frac{\partial^4\varphi}{\partial\xi^4}+3\frac{c_0}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi^2}+2h_0\frac{\partial^2\varphi}{\partial\xi\partial\tau}=0, \tag{30}$$

where $\varphi_2(\xi,\tau)$ is another unknown function whose governing equation will be obtained from the higher order perturbation expansion.

Setting $U(\xi,\ \tau)=\partial\varphi/\partial\xi$ in the last equation of (30) we obtain the following Korteweg–de Vries (KdV) equation

$$\frac{\partial U}{\partial\tau}+\frac{3}{2c_0}U\frac{\partial U}{\partial\xi}+\frac{h_0^2}{6}\frac{\partial^3U}{\partial\xi^3}=0. \tag{31}$$

To obtain the solution to $O(\epsilon^3)$ equations we introduce the solutions (26), (28) and (30) into the Eqs. (24) and (25), which yields

$$\frac{\partial^2\phi_3}{\partial z^2}+\frac{1}{24}\frac{\partial^6\varphi}{\partial\xi^6}(z^4+4h_0z^3)-\left(2\frac{\partial^4\varphi}{\partial\xi^3\partial\tau}+\frac{1}{2}\frac{\partial^4\varphi_1}{\partial\xi^4}\right)z^2+\left(9\frac{c_0}{g}\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial^3\varphi}{\partial\xi^3}+3\frac{c_0}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^4\varphi}{\partial\xi^4}-h_0\frac{\partial^4\varphi_1}{\partial\xi^4}\right.$$
$$\left.-\ 2h_0\frac{\partial^4\varphi}{\partial\xi^3\partial\tau}\right)z+2c_1(\tau)\frac{\partial^2\varphi}{\partial\xi^2}+\frac{\partial^2\varphi}{\partial\tau^2}+2\frac{\partial^2\varphi_1}{\partial\xi\partial\tau}+\frac{\partial^2\varphi_2}{\partial\xi^2}=0,$$
$$\left.\frac{\partial\phi_3}{\partial z}\right|_{z=0}-4\frac{c_0}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi\partial\tau}-3\frac{c_0}{g}\frac{\partial}{\partial\xi}\left(\frac{\partial\varphi}{\partial\xi}\frac{\partial\varphi_1}{\partial\xi}\right)+\frac{3}{2g}\left(\frac{\partial\varphi}{\partial\xi}\right)^2\frac{\partial^2\varphi}{\partial\xi^2}-2\frac{c_0}{g}\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial\varphi}{\partial\tau}-2c_0\frac{h_0^2}{g}\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial^3\varphi}{\partial\xi^3}$$
$$-c_0\frac{h_0^2}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^4\varphi}{\partial\xi^4}+h_0\frac{\partial^2\varphi_2}{\partial\xi^2}=0,$$
$$\left.\frac{\partial\phi_3}{\partial z}\right|_{z=-h_0}-h(\tau)\frac{\partial^2\varphi}{\partial\xi^2}=0. \tag{32}$$

From the solution of Eq. (32) we get

$$\begin{aligned}\phi_3 = &-\frac{1}{720}\frac{\partial^6\varphi}{\partial\xi^6}(z^6+6h_0z^5)+\frac{1}{12}\left(2\frac{\partial^4\varphi}{\partial\xi^3\partial\tau}+\frac{1}{2}\frac{\partial^4\varphi_1}{\partial\xi^4}\right)z^4\\ &-\frac{1}{6}\left(9\frac{c_0}{g}\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial^3\varphi}{\partial\xi^3}+3\frac{c_0}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^4\varphi}{\partial\xi^4}-h_0\frac{\partial^4\varphi_1}{\partial\xi^4}-2h_0\frac{\partial^4\varphi}{\partial\xi^3\partial\tau}\right)z^3\\ &-\frac{1}{2}\left[2c_1(\tau)\frac{\partial^2\varphi}{\partial\xi^2}+\frac{\partial^2\varphi}{\partial\tau^2}+2\frac{\partial^2\varphi_1}{\partial\xi\partial\tau}+\frac{\partial^2\varphi_2}{\xi^2}\right]z^2+\left[4\frac{c_0}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi\partial\tau}+3\frac{c_0}{g}\frac{\partial}{\partial\xi}\left(\frac{\partial\varphi}{\partial\xi}\frac{\partial\varphi_1}{\partial\xi}\right)\right.\\ &\left.-\frac{3}{2g}\left(\frac{\partial\varphi}{\partial\xi}\right)^2\frac{\partial^2\varphi}{\partial\xi^2}+2\frac{c_0}{g}\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial\varphi}{\partial\tau}+2c_0\frac{h_0^2}{g}\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial^3\varphi}{\partial\xi^3}+c_0\frac{h_0^2}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^4\varphi}{\partial\xi^4}-h_0\frac{\partial^2\varphi_2}{\partial\xi^2}\right]z+\varphi_3,\end{aligned} \tag{33}$$

where $\varphi_3(\xi,\tau)$ is another unknown function whose governing equation will be obtained from the higher order expansions. The use of the last boundary condition in (32) yields the following evolution equation

$$\frac{\partial^2\varphi_1}{\partial\xi\partial\tau}+\frac{3}{2c_0}\frac{\partial}{\partial\xi}\left(\frac{\partial\varphi}{\partial\xi}\frac{\partial\varphi_1}{\partial\xi}\right)+\frac{h_0^2}{6}\frac{\partial^4\varphi_1}{\partial\xi^4}=S(\varphi), \tag{34}$$

where the function $S(\varphi)$ is defined by

$$\begin{aligned}S(\varphi) = &\frac{h_0^4}{60}\frac{\partial^6\varphi}{\partial\xi^6}-\frac{h_0^2}{6}\frac{\partial^4\varphi}{\partial\xi^3\partial\tau}+\frac{5c_0h_0}{4g}\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial^3\varphi}{\partial\xi^3}+\frac{c_0h_0}{4g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^4\varphi}{\partial\xi^4}+\left[\frac{h(\tau)}{2h_0}-c_1(\tau)\right]\frac{\partial^2\varphi}{\partial\xi^2}\\ &-\frac{c_0}{gh_0}\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial\varphi}{\partial\tau}-\frac{1}{2}\frac{\partial^2\varphi}{\partial\tau^2}-2\frac{c_0}{gh_0}\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi\partial\tau}+\frac{3}{4gh_0}\left(\frac{\partial\varphi}{\partial\xi}\right)^2\frac{\partial^2\varphi}{\partial\xi^2}.\end{aligned} \tag{35}$$

Setting $V=\partial\varphi_1/\partial\xi$, the Eq. (34) reduces to the degenerate (linearized) KdV equation with a non-homogeneous term

$$\frac{\partial V}{\partial\tau}+\frac{3}{2c_0}\frac{\partial}{\partial\xi}(UV)+\frac{h_0^2}{6}\frac{\partial^3 V}{\partial\xi^3}=S(\varphi). \tag{36}$$

From the expression of $S(\varphi)$, the Eq. (35), it is seen that $S(\varphi)$ contains the unknown coefficient function $c_1(\tau)$, which is to be determined so as to remove the possible secularities that might occur. This will be investigated in the following subsection.

### *2.2. Progressive wave solution*

In this subsection we shall present the progressive wave solution to the evolution equations (31) and (36). For this purpose, we shall assume that the field variables admit a solution of the form

$$U=U(\zeta),\qquad V=V(\zeta),\qquad \zeta=\alpha(\xi-\beta\tau), \tag{37}$$

where $\alpha$ and $\beta$ are two constants to be determined from the solution. Introducing (37) into the evolution equation (31) we have

$$-\beta U'+\frac{3}{2c_0}UU'+\frac{h_0^2}{6}\alpha^2U'''=0, \tag{38}$$

where a prime denotes the differentiation of the corresponding function with respect to $\zeta$. Integrating (38) with respect to $\zeta$ and utilizing the localization conditions, i. e., $U$ and its various order derivatives vanish as $\zeta\to\pm\infty$, we have

$$-\beta U+\frac{3}{4c_0}U^2+\frac{h_0^2}{6}\alpha^2U''=0. \tag{39}$$

The conventional solitary wave solution to the Eq. (39) may be given by

$$U=a\ \mathrm{sech}^2\,\zeta,\qquad \alpha=\frac{1}{2h_0}\left(\frac{3a}{c_0}\right)^{1/2},\qquad \beta=\frac{a}{2c_0}, \tag{40}$$

where $a$ is a constant standing for the wave amplitude.

Introducing the proposed solution for $V$, Eq. (37), into (35) and (36), and integrating the result with respect to $\zeta$ we obtain

$$-\beta V+\frac{3}{2c_0}(UV)+\frac{a}{8c_0}V''=T(U), \tag{41}$$

where the function $T(U)$ is defined by

$$T(U)=\frac{3}{320}\frac{a^2}{c_0^2}U^{(4)}+\frac{a^2}{16c_0^2}U''+\frac{3a}{8c_0^2}(U')^2+\frac{3a}{16c_0^2}UU''+\left[\frac{h(\tau)}{2h_0}-c_1(\tau)-\frac{a^2}{8c_0^2}\right]U+\frac{3}{4}\frac{a}{c_0^2}U^2+\frac{1}{4c_0^2}U^3. \tag{42}$$

Introducing the solution (40) into the Eqs. (41) and (42), the following differential equation is obtained

$$V''+(12\,\mathrm{sech}^2\,\zeta-4)V=-10\frac{a^2}{c_0}\,\mathrm{sech}^6\,\zeta+12\frac{a^2}{c_0}\,\mathrm{sech}^4\,\zeta+\left[\frac{11a^2}{5c_0}+\frac{4c_0h(\tau)}{h_0}-8c_0c_1(\tau)\right]\mathrm{sech}^2\,\zeta. \tag{43}$$

Now, we shall seek a solution to Eq. (43) of the following form

$$V=A\,\mathrm{sech}^4\zeta+B\,\mathrm{sech}^2\zeta, \tag{44}$$

where $A$ and $B$ are two constants to be determined from the solution. Carrying out the derivative of $V$ we have

$$V''=-20A\mathrm{sech}^6\zeta+(16A-6B)\mathrm{sech}^4\zeta+4B\mathrm{sech}^2\zeta. \tag{45}$$

Inserting (44) and (45) into the differential equation (43) we obtain

$$-8A\mathrm{sech}^6\zeta+(12A+6B)\mathrm{sech}^4\zeta=-10\frac{a^2}{c_0}\mathrm{sech}^6\zeta+12\frac{a^2}{c_0}\mathrm{sech}^4\zeta+\left[\frac{11a^2}{5c_0}+\frac{4c_0h(\tau)}{h_0}-8c_0c_1(\tau)\right]\mathrm{sech}^2\zeta. \tag{46}$$

As seen from the Eq. (46), there is no $\mathrm{sech}^2\zeta$ term in the right hand side. In order to balance the equation, the coefficient of $\mathrm{sech}^2\zeta$ must vanish, which results in

$$c_1(\tau)=\frac{11}{40}\frac{a^2}{c_0}+\frac{h(\tau)}{2h_0}. \tag{47}$$

From the solution of the remaining equation we obtain

$$A=\frac{5}{4}\frac{a^2}{c_0},\qquad B=-\frac{a^2}{2c_0}. \tag{48}$$

Thus, in terms of the real physical entities, the final solution for $\psi$ takes the following form

$$\psi=\frac{c_0}{g}\left[a\,\mathrm{sech}^2\zeta+\epsilon\frac{a^2}{c_0}\left(\frac{5}{4}\mathrm{sech}^4\zeta-\frac{1}{2}\mathrm{sech}^2\zeta\right)\right], \tag{49}$$

where

$$\zeta=\frac{1}{2h_0}\left(\frac{3a}{c_0}\right)^{1/2}\left[\epsilon^{1/2}(x-c_0t)-\epsilon^{5/2}\frac{9}{40}\frac{a^2}{c_0}x+\frac{\epsilon}{2h_0}\int_0^{\epsilon^{3/2}x}h(s)\mathrm{d}s\right]. \tag{50}$$

As seen from Eq. (50), the phase function $\zeta$ is not linear in the variables $x$ and $t$ anymore; it is rather nonlinear in $x$. As a result of this, the speed of the propagation will be a variable in $x$. From Eq. (50), the speed may be given by

$$v=\frac{c_0}{1+\epsilon^2\left[\frac{h(\epsilon^{3/2}x)}{2h_0}-\frac{9}{40}\frac{a^2}{c_0}\right]}. \tag{51}$$

This shows that the variation of the depth of water along the $x$ axis will also change the speed of propagation.

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